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Vector Variational Formulation of Electromagnetic Fields in Anisotropic Media

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Abstract—**Maxwell's equations can be cast into a basic differential operator equation, the curlcurl equation, which lends itself easily to variational treatment. Various forms of this equation are associated with problems of practical importance. The formulation includes the treatment of loss-free anisotropic media. The boundary conditions associated with electromagnetic-field problems are treated in detail and the uniqueness of the solution is discussed. A functional is derived for the curlcurl equation in Cartesian and cylindrical coordinates.**

I. Introduction

DUE TO the broad variety of practical applications of waveguides, resonators, and other microwave devices, the development of methods to solve the associated electromagnetic-field problems has received a great deal of attention in the past two decades. Such electromagnetic boundary value problems, with the exception of isotropic waveguides, require a formulation in which the electric and magnetic fields are treated as vector quantities. In recent years, a variety of methods for the solution of homogeneous isotropic waveguide problems appeared in the literature; these have been reviewed by Wexler [1], by Davies [2], and by Ng [3]. With a few exceptions [4]–[9], [29], the tendency in recent years was to formulate the inhomogeneous isotropic waveguide problem in terms of the longitudinal electric (E_z) and magnetic (H_z) field components [10]–[18].

As noted by Wexler [1] in 1969, there have been many proponents and only a few attempts to formulate electromagnetic-field problems in terms of all three components of the field vectors. Among the attempts one must mention Harrington's well-known monograph [4] and Gupta's doctoral dissertation [5] on field solution in resonant cavities filled with an inhomogeneous anisotropic medium. The moment method employed by these authors is essentially a projective method in which the field components in a cavity or waveguide are expanded in terms of the field components of the empty cavity or waveguide modes.

In 1967 Hannaford [10] proposed an extension of his variational/finite difference method for homogeneous isotropic waveguides to plasma- and ferrite-filled waveguides. Hannaford's proposal involves only the longitudinal field components. For inhomogeneous media, the resulting coefficient matrix in Hannaford's formulation becomes indefinite above the 45° "air-line" on the dispersion diagram. Since 1967, this shortcoming of two-component formulations has reoccurred in a number of other finite-difference and finite-element variational methods [11]–[17]. Hannaford dismissed Berk's often quoted variational expressions which were published in 1956 [6] as being more complicated than the E_z–H_z formulation. Berk derived three- and six-component vector variational expressions in the form of Rayleigh quotients for the resonance frequencies of a resonator filled with loss-free, anisotropic, homogeneous or inhomogeneous media.

The only three-component vector variational formulation for electromagnetic-field problems appearing in recent years is due to English and Young [7]. They select the $\boldsymbol{E}$-field formulation over $\boldsymbol{H}$ on the basis of the number of Dirichlet boundary conditions to be satisfied. The authors list the advantages of the three-component vector formulation as reduced matrix size and denser coefficient matrices in comparison with the six-component formulation given by English in his doctoral dissertation [8] and in two papers by English [9], [23] which appeared in 1971. However,

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they find that the resulting matrix elements are more complicated to calculate, the guide-wall boundary conditions on the trial functions are more restrictive, and the imposition of continuity constraints on the trial field components is less straightforward. English and Young apply their method to inhomogeneously filled isotropic parallel-plate waveguides and to rectangular waveguides. Unfortunately, in their three-component formulation the condition $\boldsymbol{n} \times \boldsymbol{E} = 0$ must be satisfied exactly by the trial functions, so that waveguide shapes other than circular or rectangular cannot be treated.

It is evident from the foregoing survey of the literature that the need for a general three-component vector variational formulation of loss-free, bounded electromagnetic-field problems has long been recognized. This semitutorial paper contains a unified three-component vector variational formulation of electromagnetic fields not only for isotropic media, but for homogeneous and inhomogeneous anisotropic waveguide and resonator problems as well. The variational expressions derived in this paper are independent of the method chosen for discretization (e.g., finite differences, finite elements). The discussion on boundary conditions and uniqueness highlights the advantages as well as some of the shortcomings of vector variational formulations in general and the vector variational formulation of the inhomogeneous waveguide problem in particular. It is shown, for example, that the occurrence of nonphysical (spurious) solutions in vector variational formulations is due to a larger than expected set of natural boundary conditions.

II. The Curlcurl Equation

Consider Maxwell's curl equations for time-harmonic fields [19]

$$\text{curl}\, \boldsymbol{E} = -j\omega \boldsymbol{B} \tag{2.1}$$

$$\text{curl}\, \boldsymbol{H} = +j\omega \boldsymbol{D} + \boldsymbol{J}. \tag{2.2}$$

The vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric- and magnetic-field intensities and the vectors $\boldsymbol{D}$ and $\boldsymbol{B}$ are the electric- and magnetic-flux densities, respectively. $\boldsymbol{J}$ denotes current density and it includes impressed currents ($\boldsymbol{J}_i$) as well as induced conduction currents ($\boldsymbol{J}_c$).

Let $\hat{\mu}$ and $\hat{\varepsilon}$ represent the tensor permeability and tensor permittivity, respectively. By substituting constitutive relationships for linear media in (2.1) and (2.2), taking curl of both sides, and then substituting for curl $\boldsymbol{H}$ from (2.2) and for curl $\boldsymbol{E}$ from (2.1), the following equations are obtained for nonconductive media:

$$\text{curl}\, (\hat{\mu}^{-1}\, \text{curl}\, \boldsymbol{E}) - \omega^2 \hat{\varepsilon} \boldsymbol{E} = -j\omega \boldsymbol{J}_i \tag{2.3}$$

$$\text{curl}\, (\hat{\varepsilon}^{-1}\, \text{curl}\, \boldsymbol{H}) - \omega^2 \hat{\mu} \boldsymbol{H} = \text{curl}\, (\hat{\varepsilon}^{-1} \boldsymbol{J}_i). \tag{2.4}$$

In a conductive medium, the conduction current $\boldsymbol{J}_c$ causes a magnetic field $\boldsymbol{H}$ to appear. If $\boldsymbol{J}_c$ is the only current flowing, then (2.2) gives

$$\text{curl}\, \boldsymbol{H} = \boldsymbol{J}_c \tag{2.5}$$

which can be rewritten in terms of the magnetic vector potential $\boldsymbol{A}$ as

$$\text{curl}\, (\hat{\mu}^{-1}\, \text{curl}\, \boldsymbol{A}) = \boldsymbol{J}_c. \tag{2.6}$$

The aim of the following sections of this paper is to present a unified variational formulation for problems involving the Maxwell equations for nonconductive media expressed as (2.3) or (2.4) and the magnetic vector potential expressed as (2.6) in bounded regions. These three equations are special cases of the following general equation depending on the interpretation of $\hat{p}$, $\hat{q}$, $\boldsymbol{V}$, and $\boldsymbol{g}$:

$$\text{curl}\, (\hat{p}\, \text{curl}\, \boldsymbol{V}) - \omega^2 \hat{q} \boldsymbol{V} = -\boldsymbol{g}. \tag{2.7}$$

The differential operator in this equation is self-adjoint provided that $\hat{p}$ and $\hat{q}$ are Hermitian and therefore lends itself readily to a variational formulation [20].

III. Functional Formulation

According to the Minimum Theorem [27] the vector function $\boldsymbol{V}$ which satisfies the curlcurl equation (2.7) minimizes an energy-related functional given by

$$F(\boldsymbol{v}) = \langle \text{curl}\, (\hat{p}\, \text{curl}\, \boldsymbol{v}), \boldsymbol{v} \rangle - \omega^2 \langle \hat{q}\boldsymbol{v}, \boldsymbol{v} \rangle + \langle \boldsymbol{g}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, \boldsymbol{g} \rangle. \tag{3.1}$$

In view of the fact that the electric- and magnetic-field intensities vary harmonically with time, and therefore have both magnitude and phases, the following inner product should be used

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \iiint_{\Omega} (\boldsymbol{b}^* \cdot \boldsymbol{a})\, dU. \tag{3.2}$$

The asterisk here denotes complex conjugate. With this definition of inner product, the functional can be rewritten as

$$\begin{aligned} F(\boldsymbol{v}) = &\iiint_{\Omega} [\boldsymbol{v}^* \cdot \text{curl}\, (\hat{p}\, \text{curl}\, \boldsymbol{v})]\, dU \\ &- \omega^2 \iiint_{\Omega} (\boldsymbol{v}^* \cdot \hat{q}\boldsymbol{v})\, dU \\ &+ \iiint_{\Omega} (\boldsymbol{v}^* \cdot \boldsymbol{g})\, dU + \iiint_{\Omega} (\boldsymbol{g}^* \cdot \boldsymbol{v})\, dU. \end{aligned} \tag{3.3}$$

Consider now the following vector identity:

$$\text{div}\, (\boldsymbol{a} \times \boldsymbol{b}) = (\text{curl}\, \boldsymbol{a}) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \text{curl}\, \boldsymbol{b}. \tag{3.4}$$

Integrating both sides and then applying the divergence theorem to the left-hand side yields

$$\oiint_{\Gamma} (\boldsymbol{a} \times \boldsymbol{b}) \cdot d\boldsymbol{S} = \iiint_{\Omega} [(\text{curl}\, \boldsymbol{a}) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \text{curl}\, \boldsymbol{b}]\, dU. \tag{3.5}$$

With $\boldsymbol{a}$ replaced by $\boldsymbol{v}^*$ and $\boldsymbol{b}$ replaced by ($\hat{p}$ curl $\boldsymbol{v}$) one obtains

$$\begin{aligned} &\iiint_{\Omega} [\boldsymbol{v}^* \cdot \text{curl}\, (\hat{p}\, \text{curl}\, \boldsymbol{v})]\, dU \\ &= \iiint_{\Omega} [\text{curl}\, (\boldsymbol{v}^*) \cdot (\hat{p}\, \text{curl}\, \boldsymbol{v})]\, dU \\ &\quad - \oiint_{\Gamma} [\boldsymbol{v}^* \times (\hat{p}\, \text{curl}\, \boldsymbol{v})] \cdot \boldsymbol{n}\, dS \end{aligned} \tag{3.6}$$

which is merely the application of Green's first identity in vector form. Now substituting (3.6) into (3.3) one obtains

$$F(\boldsymbol{v}) = \iiint_\Omega [(\text{curl } \boldsymbol{v})^* \cdot (\hat{p} \text{ curl } \boldsymbol{v})]\, dU - \omega^2 \iiint_\Omega (\boldsymbol{v}^* \cdot \hat{q}\boldsymbol{v})\, dU + \iiint_\Omega (\boldsymbol{v}^* \cdot \boldsymbol{g} + \boldsymbol{g}^* \cdot \boldsymbol{v})\, dU - \oint\!\!\oint_\Gamma [\boldsymbol{v}^* \times (\hat{p} \text{ curl } \boldsymbol{v})] \cdot \boldsymbol{n}\, dS. \quad (3.7)$$

By denoting the components of the vectors $\boldsymbol{g}$, $\boldsymbol{v}$, and curl $\boldsymbol{v}$ by g_i, v_i, and $(\text{curl } \boldsymbol{v})_i$, $i = 1,2,3$, and the components of the tensors $\hat{p}$ and $\hat{q}$ by $p_{i,j}$ and $q_{i,j}$, $i,j = 1,2,3$, respectively, one can rewrite the functional in the following form:

$$F(\boldsymbol{v}) = \sum_{i=1}^{3} \iiint_\Omega \Big[p_{i,i}|(\text{curl } \boldsymbol{v})_i|^2 + p_{i,i+1}(\text{curl } \boldsymbol{v})_i^*(\text{curl } \boldsymbol{v})_{i+1} + p_{i+1,i}(\text{curl } \boldsymbol{v})_i(\text{curl } \boldsymbol{v})_{i+1}^* - \omega^2(q_{i,i}|v_i|^2 + q_{i,i+1}v_i^* v_{i+1} + q_{i+1,i}v_i v_{i+1}^*) + v_i^* g_i + g_i^* v_i\Big]\, dU + \sum_{i=1}^{3} \oint\!\!\oint_\Gamma \Big[\mathbf{1}_i \sum_{j=1}^{3} (p_{i+1,j}v_{i+2}^* - p_{i+2,j}v_{i+1}^*)(\text{curl } \boldsymbol{v})_j\Big] \cdot \boldsymbol{n}\, dS. \quad (3.8)$$

The subscripts in (3.8) are cyclic modulo 3. The unit vector in the ith coordinate direction is denoted by $\mathbf{1}_i$. The volume integral is defined over some volume Ω bounded by a surface Γ. The unit vector $\boldsymbol{n}$ is outward normal everywhere to the surface Γ.

For loss-free passive media the material property tensors $\hat{p}$ and $\hat{q}$ are always Hermitian. Consequently, the following relations hold true [20]:

$$q_{i,i+1}v_i^* v_{i+1} + q_{i+1,i}v_i v_{i+1}^* = 2 \text{ Re } (q_{i+1,i}v_{i+1}^* v_i) \quad (3.9)$$

$$p_{i,i+1}(\text{curl } \boldsymbol{v})_i^*(\text{curl } \boldsymbol{v})_{i+1} + p_{i+1,i}(\text{curl } \boldsymbol{v})_i(\text{curl } \boldsymbol{v})_{i+1}^* = 2 \text{ Re } [p_{i+1,i}(\text{curl } \boldsymbol{v})_{i+1}^*(\text{curl } \boldsymbol{v})_i]. \quad (3.10)$$

Therefore, the functional in (3.8) can be rewritten in the following way:

$$F(\boldsymbol{v}) = \sum_{i=1}^{3} \iiint_\Omega \Big\{p_{i,i}|(\text{curl } \boldsymbol{v})_i|^2 + 2 \text{ Re } [p_{i+1,i}(\text{curl } \boldsymbol{v})_{i+1}^*(\text{curl } \boldsymbol{v})_i] - \omega^2[q_{i,i}|v_i|^2 + 2 \text{ Re } (q_{i+1,i}v_{i+1}^* v_i)] + 2 \text{ Re } (g_i v_i^*)\Big\}\, dU + \sum_{i=1}^{3} \oint\!\!\oint_\Gamma \Big[\mathbf{1}_i \sum_{j=1}^{3} (p_{i+1,j}v_{i+2}^* - p_{i+2,j}v_{i+1}^*)(\text{curl } \boldsymbol{v})_j\Big] \cdot \boldsymbol{n}\, dS. \quad (3.11)$$

At this point one could easily ask: How does one know if the functional (3.11) is correct? If the integrand of the volume integral part of the functional $F(\boldsymbol{v})$ is denoted by L, then according to the calculus of variations, the first variation of F will be zero provided that the following equations are satisfied:

$$\sum_{j=1}^{3} \left\{\frac{\partial^2 L}{\partial a_j\, \partial \left(\frac{\partial v_i}{\partial a_j}\right)}\right\} = \frac{\partial L}{\partial v_i}, \qquad i = 1,2,3. \quad (3.12)$$

These equations are referred to as the Euler equations associated with the Lagrangian L [21]. The a_j represent spatial coordinates. It has been verified that (3.12) reduces (3.11) to the curlcurl equation.

IV. Boundary Conditions

Let $\boldsymbol{v}$ represent the electric-field intensity $\boldsymbol{E}$. Then

$$\hat{p} \text{ curl } \boldsymbol{v} = \hat{\mu}^{-1} \text{ curl } \boldsymbol{E} = -j\omega \boldsymbol{H} \quad (4.1)$$

and the surface integral term in the functional (3.7) can be written as

$$-\oint\!\!\oint_\Gamma [\boldsymbol{v}^* \times (\hat{p} \text{ curl } \boldsymbol{v})] \cdot \boldsymbol{n}\, dS = j\omega \iint_\Gamma (\boldsymbol{E}^* \times \boldsymbol{H}) \cdot \boldsymbol{n}\, dS. \quad (4.2)$$

The cross product $\boldsymbol{E}^* \times \boldsymbol{H}$ is the well-known complex Poynting vector representing the density of power flux. Therefore, the surface integral represents the net power flow across the boundary surface. If the boundary Γ is a perfect conductor, then no energy is transferred and the Poynting vector is tangential everywhere to the surface. Mathematically, this idea is expressed by the equation

$$[\boldsymbol{v}^* \times (\hat{p} \text{ curl } \boldsymbol{v})] \cdot \boldsymbol{n} = 0. \quad (4.3)$$

The boundary conditions that are implicitly enforced by leaving out the surface integral from the functional, i.e., that correspond to the choice given by (4.3), are called the natural boundary conditions of the functional [22]. It is a well-known property of scalar triple products that they remain unchanged under a cyclic permutation of three vectors. Thus one can write the following equalities:

$$[\boldsymbol{v}^* \times (\hat{p} \text{ curl } \boldsymbol{v})] \cdot \boldsymbol{n} = [(\hat{p} \text{ curl } \boldsymbol{v}) \times \boldsymbol{n}] \cdot \boldsymbol{v}^* = (\boldsymbol{n} \times \boldsymbol{v}^*) \cdot (\hat{p} \text{ curl } \boldsymbol{v}). \quad (4.4)$$

It will now be shown that the boundary conditions implicit in (4.3) are exactly the same as those commonly encountered in electromagnetic-field problems.

Let $\boldsymbol{v}$ represent the electric field $\boldsymbol{E}$ again. Then by (4.4) it is obvious that (4.3) will be satisfied whenever

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{n} \times \hat{\varepsilon}^{-1} \text{ curl } \boldsymbol{H} = 0 \quad \text{on } \Gamma. \quad (4.5)$$

Now, let $\boldsymbol{v}$ represent the magnetic field $\boldsymbol{H}$. Then, again by virtue of (4.4) it can be seen that (4.3) will be satisfied whenever (4.5) is true. The boundary condition expressed in (4.5) is the one commonly used for perfect electric conductors [19]. It merely states that the electric-field intensity vector $\boldsymbol{E}$ must be normal everywhere at the boundary, without specifying the magnitude of $\boldsymbol{E}$.

The boundary condition given by

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{n} \times \hat{\mu}^{-1} \operatorname{curl} \boldsymbol{E} = 0 \quad \text{on} \quad \Gamma \tag{4.6}$$

is equally correct and satisfies (4.3), but it is only meaningful if one accepts the idea of perfect magnetic conductor. This is defined as a material for which $\boldsymbol{H}$ must be normal everywhere at its surface.[1]

The boundary conditions (4.5) and (4.6) can also be obtained in terms of the magnetic vector potential $\boldsymbol{A}$. For perfect electric conductors one can write

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{n} \times \boldsymbol{A} = 0 \quad \text{on} \quad \Gamma \tag{4.7}$$

while for perfect magnetic conductors one obtains

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{n} \times \hat{\mu}^{-1} \operatorname{curl} \boldsymbol{A} = 0 \quad \text{on} \quad \Gamma. \tag{4.8}$$

The conditions of the type $(\hat{p} \operatorname{curl} \boldsymbol{v}) \times \boldsymbol{n} = 0$ result in three impedance-type boundary conditions; i.e., constraints on the normal derivatives of each component of $\boldsymbol{v}$ [20]. These are natural boundary conditions of the functional when the surface integral is set to zero and will be automatically satisfied by the function which minimizes the functional. In other words, one does not need to restrict the set from which the function $\boldsymbol{v}$ is taken. Boundary conditions such as the ones implicit in (4.7) have, of course, to be taken care of explicitly.

One question in connection with boundary conditions remains unanswered: Are they sufficient to guarantee a unique solution? The answer to this question is not at all obvious since one is dealing with vector quantities. At least one well-known textbook on electromagnetic theory states that two conditions must be specified for a vector function [28].

V. Uniqueness

It will now be shown that the boundary conditions given in (4.5)–(4.8) for perfect electric or magnetic conductors do indeed guarantee unique solutions to the curlcurl equation (2.7), [24]. Suppose that two distinct solutions exist for the same boundary value problem and denote them by $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. Electromagnetically, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ could be either electric-field intensity vectors or magnetic-field vectors. It is required that the curls of the two solutions be equal so that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ both have the same volume sources. Due to the linear nature of the curl operator the difference solution $\boldsymbol{V}_d = \boldsymbol{V}_1 - \boldsymbol{V}_2$ also satisfies (2.7), but with a vanishing source term. Moreover, the curl of $\boldsymbol{V}_d$ is zero.

The energy norm of the vector field $\boldsymbol{V}$ will be defined by the following integral

$$\|\boldsymbol{V}\| = \left[\iiint \boldsymbol{V}^* \cdot \hat{q} \boldsymbol{V} \, dU \right]^{1/2} \tag{5.1}$$

where the integration is over a volume Ω bounded by a surface Γ. The norm of $\boldsymbol{V}$ as given by (5.1) is a number assigned to $\boldsymbol{V}$ which is in the energy sense a measure of the magnitude of $\boldsymbol{V}$. The vector function $\boldsymbol{V}$ belongs to a linear space S. The norm given in (5.1) is valid provided that the following conditions are met:

1) $\|\boldsymbol{a}\| \geq 0, \quad \boldsymbol{a} \in S$;
2) $\|c\boldsymbol{a}\| = |c|\|\boldsymbol{a}\|$, where c is any real number;
3) $\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$ (triangle inequality), $\boldsymbol{b} \in S$;
4) $\|\boldsymbol{a}\| = 0$, implies $\boldsymbol{a} \equiv 0$.

If the 3-by-3 nonsingular Hermitian matrix representing the material property tensor $\hat{q}$ is positive definite, then the Hermitian form $[V]^*[q][V]$ is strictly positive for all nontrivial $[V]$. This is the only requirement needed to meet the four aforementioned conditions. Permittivity and permeability tensors of passive media are all positive-definite 3-by-3 matrices.

One would like to find the conditions under which the square of the energy norm of $\boldsymbol{V}_d$ vanishes. If the energy norm is zero then by property 4) $\boldsymbol{V}_d$ itself will be zero "almost everywhere."[2] By substituting for $\hat{q}\boldsymbol{V}_d$ from the curlcurl equation and then using Green's first identity in vector form [see (3.6)], the energy norm of $\boldsymbol{V}_d$ can be transformed as follows:

$$\begin{aligned} &\iiint \boldsymbol{V}_d^* \cdot (\hat{q} \boldsymbol{V}_d) \, dU \\ &= (1/\omega^2) \iiint \boldsymbol{V}_d^* \cdot [\operatorname{curl} (\hat{p} \operatorname{curl} \boldsymbol{V}_d)] \, dU \\ &= (1/\omega^2) \iiint_\Omega [\operatorname{curl} (\boldsymbol{V}_d^*) \cdot (\hat{p} \operatorname{curl} \boldsymbol{V}_d)] \, dU \\ &\quad - (1/\omega^2) \oiint_\Gamma [\boldsymbol{V}_d^* \times (\hat{p} \operatorname{curl} \boldsymbol{V}_d)] \cdot \boldsymbol{n} \, dS. \end{aligned} \tag{5.2}$$

The volume integral over Ω vanishes because the curl of $\boldsymbol{V}_d$ is zero within the volume. In order to make the surface integral vanish one requires that the integrand be zero

$$[\boldsymbol{V}_d^* \times (\hat{p} \operatorname{curl} \boldsymbol{V}_d)] \cdot \boldsymbol{n} = 0. \tag{5.3}$$

This is true whenever either

$$\boldsymbol{n} \times \boldsymbol{V}_d = 0 \tag{5.4}$$

or

$$\boldsymbol{n} \times (\hat{p} \operatorname{curl} \boldsymbol{V}_d) = 0. \tag{5.5}$$

Obviously, if $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ both satisfy either of these conditions at the boundary surface, then $\boldsymbol{V}_d$ also satisfies them and (5.2) is equal to zero. Therefore, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are one and the same unique solution of the curlcurl equation. Note that the proof breaks down when the frequency ω is zero.

Unfortunately, the energy norm of $\boldsymbol{V}_d$ may also vanish when neither (5.4) nor (5.5) is satisfied. In such cases, the surface integral over Γ vanishes over the boundary surface as a whole. Physically, such a situation requires a surface through which energy transfer is possible, but the net energy transferred must be zero. It is this condition which causes "spurious," "nonphysical" modes to appear in variational solution of waveguide problems [11], [13]–[17].

[1] In many practical problems where symmetry exists, the planes of symmetry will behave like perfect magnetic conductors.

[2] "Almost everywhere" implies everywhere except on a denumerable subset of Ω such as the surface Γ.

To overcome the problem of spurious solutions, one must ensure that the surface integral in the functional (3.11) vanishes only under conditions of the type (5.4) or (5.5). In a three-component $\boldsymbol{H}$-field formulation this amounts to asking that $\boldsymbol{B}$ be everywhere tangential to perfect electric conductors. The enforcement of the condition $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ guarantees that $\boldsymbol{n} \cdot (\text{curl } \boldsymbol{E}) = 0$; i.e., that curl $\boldsymbol{E}$ will be tangential to the boundary surface. This in turn guarantees that $\boldsymbol{E}$ will be normal to the boundary, and hence that $\boldsymbol{n} \times \hat{\varepsilon}^{-1}$ curl $\boldsymbol{H} = 0$. In a three-component $\boldsymbol{E}$-field formulation the condition $\boldsymbol{n} \cdot \boldsymbol{D} = 0$ must be enforced at perfect magnetic conductors. It is not more difficult to enforce the conditions $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ or $\boldsymbol{n} \cdot \boldsymbol{D} = 0$ than it is to enforce the condition $\boldsymbol{n} \times \boldsymbol{E} = 0$ or $\boldsymbol{n} \times \boldsymbol{H} = 0$.

The situation is quite different for the magnetic vector potential $\boldsymbol{A}$. The boundary conditions given in (4.7) and (4.8) in terms of $\boldsymbol{A}$ for perfect electric and magnetic conductors, respectively, guarantee that the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ derived from the solution of (2.6) will be unique. However, unless the divergence of $\boldsymbol{A}$ is somehow fixed, $\boldsymbol{A}$ itself is not unique. If the Coulomb convention is adopted, the divergence of $\boldsymbol{A}$ will be zero and (2.6) will be reduced to the vector Poisson equation. Even so, the solution will not be unique if only the boundary condition $\boldsymbol{n} \times \hat{\mu}^{-1}$ curl $\boldsymbol{A} = 0$ is applied to all parts of the boundary.

VI. Summary and the Explicit Forms of the Functional

If a region of space is bounded by a perfect electric conductor with no magnetic currents flowing on its surface, i.e., if $\boldsymbol{n} \times \boldsymbol{E}$ is zero, then (2.3) can be solved for the vector $\boldsymbol{E}$ by using the boundary condition $\boldsymbol{n} \times \boldsymbol{E} = 0$. The corresponding vector $\boldsymbol{H}$ can be obtained for the same problem by solving (2.4) with the boundary condition $\boldsymbol{n} \times \hat{\varepsilon}^{-1}$ curl $\boldsymbol{H} = 0$. The electric current induced on the surface of the perfect electric conductor can be obtained by evaluating $\boldsymbol{n} \times \boldsymbol{H}$. The induced electric surface charge density is given by $\boldsymbol{n} \cdot (\hat{\varepsilon}\boldsymbol{E})$.

If the boundary of the region behaves like a perfect magnetic conductor with no electric current flowing on its surface, i.e., if $\boldsymbol{n} \times \boldsymbol{H}$ is zero, then (2.4) can be solved for the magnetic-field vector $\boldsymbol{H}$ by using the boundary condition $\boldsymbol{n} \times \boldsymbol{H} = 0$. The corresponding vector $\boldsymbol{E}$ can be obtained for the same problem by solving (2.3) with the boundary condition $\boldsymbol{n} \times \hat{\mu}^{-1}$ curl $\boldsymbol{E}$. The magnetic current induced on the surface of the perfect magnetic conductor is given by $\boldsymbol{n} \times \boldsymbol{E}$. The induced fictitious magnetic surface charge density is given by $\boldsymbol{n} \cdot (\hat{\mu}\boldsymbol{H})$.

If the boundary consists partly of a perfect electric conductor and partly of a perfect magnetic conductor, then the vector $\boldsymbol{E}$ can be obtained by solving (2.3) with the boundary condition $\boldsymbol{n} \times \boldsymbol{E} = 0$ on the electric conductor and with the boundary condition $\boldsymbol{n} \times \hat{\mu}^{-1}$ curl $\boldsymbol{E} = 0$ on the magnetic conductor. The corresponding vector $\boldsymbol{H}$ is obtained by solving (2.4) with the boundary condition $\boldsymbol{n} \times \hat{\varepsilon}^{-1}$ curl $\boldsymbol{H} = 0$ where the electric conductor is located and with the boundary condition $\boldsymbol{n} \times \boldsymbol{H} = 0$ where the magnetic conductor is found.

The solutions are unique in all of the previous cases.

For an abrupt discontinuity in the permittivity $\hat{\varepsilon}$ in an inhomogeneous medium there is an abrupt change in the electric field $\boldsymbol{E}$ as well. In such cases, it is advantageous to solve for the magnetic field from (2.4). Similarly, for an inhomogeneous medium with discontinuities in the permeability $\hat{\mu}$, $\boldsymbol{H}$ displays discontinuities and it is easier to solve for $\boldsymbol{E}$ from (2.3) than for $\boldsymbol{H}$ from (2.4). The simultaneous occurrence of both types of inhomogeneities is rare.

The solution of the curlcurl equation is achieved by minimizing the associated functional. The three impedance-type boundary conditions implicit in $(\hat{p} \text{ curl } \boldsymbol{v}) \times \boldsymbol{n} = 0$ are natural if the surface integral is neglected from the functional. For the boundary condition $\boldsymbol{n} \times \boldsymbol{v} = 0$, which must be taken care of explicitly, the surface integral still vanishes. Without the surface integral, the functional given in (3.11) takes on the following form in rectangular coordinates (x,y,z):

$$
\begin{aligned}
F(\boldsymbol{v}) = \iiint \mathrm{Re}\Bigg[& p_{xx}\left(\frac{\partial v_z}{\partial y}\frac{\partial v_z^*}{\partial y} - \frac{\partial v_z}{\partial y}\frac{\partial v_y^*}{\partial z} - \frac{\partial v_z^*}{\partial y}\frac{\partial v_y}{\partial z} + \frac{\partial v_y}{\partial z}\frac{\partial v_y^*}{\partial z}\right) \\
& + p_{yy}\left(\frac{\partial v_x}{\partial z}\frac{\partial v_x^*}{\partial z} - \frac{\partial v_x}{\partial z}\frac{\partial v_z^*}{\partial x} - \frac{\partial v_x^*}{\partial z}\frac{\partial v_z}{\partial x} + \frac{\partial v_z}{\partial x}\frac{\partial v_z^*}{\partial x}\right) \\
& + p_{zz}\left(\frac{\partial v_y}{\partial x}\frac{\partial v_y^*}{\partial x} - \frac{\partial v_y}{\partial x}\frac{\partial v_x^*}{\partial y} - \frac{\partial v_y^*}{\partial x}\frac{\partial v_x}{\partial y} + \frac{\partial v_x}{\partial y}\frac{\partial v_x^*}{\partial y}\right) \\
& + 2p_{yx}\left(\frac{\partial v_z}{\partial y}\frac{\partial v_x^*}{\partial z} - \frac{\partial v_y}{\partial z}\frac{\partial v_x^*}{\partial z} - \frac{\partial v_z}{\partial y}\frac{\partial v_z^*}{\partial x} + \frac{\partial v_y}{\partial z}\frac{\partial v_z^*}{\partial x}\right) \\
& + 2p_{zx}^*\left(\frac{\partial v_z^*}{\partial y}\frac{\partial v_y}{\partial x} - \frac{\partial v_y^*}{\partial z}\frac{\partial v_y}{\partial x} - \frac{\partial v_z^*}{\partial y}\frac{\partial v_x}{\partial y} + \frac{\partial v_y^*}{\partial z}\frac{\partial v_x}{\partial y}\right) \\
& + 2p_{zy}\left(\frac{\partial v_x}{\partial z}\frac{\partial v_y^*}{\partial x} - \frac{\partial v_z}{\partial x}\frac{\partial v_y^*}{\partial x} - \frac{\partial v_x}{\partial z}\frac{\partial v_x^*}{\partial y} + \frac{\partial v_z}{\partial x}\frac{\partial v_x^*}{\partial y}\right) \\
& - \omega^2(q_{xx}v_xv_x^* + q_{yy}v_yv_y^* + q_{zz}v_zv_z^* + 2q_{yx}v_xv_y^* \\
& + 2q_{zx}^*v_zv_x^* + 2q_{zy}v_yv_z^*) \\
& + 2(g_xv_x^* + g_yv_y^* + g_zv_z^*)\Bigg]\, dx\, dy\, dz. \qquad (6.1)
\end{aligned}
$$

In cylindrical coordinates (r,θ,z), the functional (3.11) is given by

$$\begin{aligned}
F(v) = \iiint \mathrm{Re}\Bigg[& p_{rr}\left(\frac{\partial v_z}{r\,\partial\theta}\frac{\partial v_z^*}{\partial\theta} - \frac{\partial v_z}{\partial\theta}\frac{\partial v_\theta^*}{\partial z} - \frac{\partial v_z^*}{\partial\theta}\frac{\partial v_\theta}{\partial z} + r\frac{\partial v_\theta}{\partial z}\frac{\partial v_\theta^*}{\partial z}\right) \\
& + p_{\theta\theta} r\left(\frac{\partial v_r}{\partial z}\frac{\partial v_r^*}{\partial z} - \frac{\partial v_r}{\partial z}\frac{\partial v_z^*}{\partial r} - \frac{\partial v_r^*}{\partial z}\frac{\partial v_z}{\partial r} + \frac{\partial v_z}{\partial r}\frac{\partial v_z^*}{\partial r}\right) \\
& + p_{zz}\left(r\frac{\partial v_\theta}{\partial r}\frac{\partial v_\theta^*}{\partial r} + v_\theta^*\frac{\partial v_\theta}{\partial r} + v_\theta\frac{\partial v_\theta^*}{\partial r} + \frac{v_\theta v_\theta^*}{r} - \frac{\partial v_r}{\partial\theta}\frac{\partial v_\theta^*}{\partial r} - \frac{\partial v_r^*}{\partial\theta}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\frac{\partial v_r^*}{\partial\theta} - \frac{v_\theta^*}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_r}{r\,\partial\theta}\frac{\partial v_r^*}{\partial\theta}\right) \\
& + 2p_{\theta r}\left(\frac{\partial v_z}{\partial\theta}\frac{\partial v_r^*}{\partial z} - r\frac{\partial v_\theta}{\partial z}\frac{\partial v_r^*}{\partial z} - \frac{\partial v_z}{\partial\theta}\frac{\partial v_z^*}{\partial r} + r\frac{\partial v_\theta}{\partial z}\frac{\partial v_z^*}{\partial r}\right) \\
& + 2p_{zr}^*\left(\frac{\partial v_z^*}{\partial\theta}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z^*}{\partial\theta} - r\frac{\partial v_\theta^*}{\partial z}\frac{\partial v_\theta}{\partial r} - v_\theta\frac{\partial v_\theta^*}{\partial z} - \frac{\partial v_z^*}{r\,\partial\theta}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta^*}{\partial z}\frac{\partial v_r}{\partial\theta}\right) \\
& + 2p_{z\theta}\left(r\frac{\partial v_r}{\partial z}\frac{\partial v_\theta^*}{\partial r} + v_\theta^*\frac{\partial v_r}{\partial z} - v_\theta^*\frac{\partial v_z}{\partial r} - r\frac{\partial v_z}{\partial r}\frac{\partial v_\theta^*}{\partial r} - \frac{\partial v_r}{\partial z}\frac{\partial v_r^*}{\partial\theta} - \frac{\partial v_z}{\partial r}\frac{\partial v_r^*}{\partial\theta}\right) \\
& - \omega^2 r(q_{rr}v_r v_r^* + q_{\theta\theta}v_\theta v_\theta^* + q_{zz}v_z v_z^* + 2q_{\theta r}v_r v_\theta^* + 2q_{zr}^* v_z v_r^* + 2q_{z\theta}v_\theta v_z^*) \\
& + 2r(g_r v_r^* + g_\theta v_\theta^* + g_z v_z^*)\Bigg]\, dr\, dz\, d\theta. \qquad (6.2)
\end{aligned}$$

Notice the factor $1/r$ in some of the terms in the integrand of (6.2). The singularity at $r = 0$ is potentially troublesome as far as the integration is concerned. However, the limit of the terms with the $1/r$ singularity as r tends to zero is indeterminate; this suggests that l'Hospital's rule can be applied. Thus the limits of these quantities are finite, suggesting that the singularity can be integrated [20], [25], [26], [30]. Notice also that unlike in E_z–H_z formulations [10]–[18], there are no singularities with respect to the material property tensors in the functionals derived here. Moreover, it is evident that the integrands of (6.1) and (6.2) are real quantities provided that the tensors are Hermitian.

VII. Conclusions

A unified three-component vector variational formulation has been presented for time-harmonic electromagnetic fields in loss-free, anisotropic media. An energy-related functional has been derived for the curlcurl equation and the associated natural boundary conditions have been examined in detail in the context of uniqueness. The apparent advantages of the formulation are the following.

1) Generality: It is valid for homogeneous or inhomogeneous, isotropic or anisotropic, loss-free media.

2) The impedance-type boundary conditions at perfect conductors are natural boundary conditions. The derived functional is suitable both for finite-difference and for finite-element discretization without special restrictions on the trial functions.

3) The formulation does not give rise to singularities with respect to the material properties of the medium such as encountered in E_z–H_z formulations for inhomogeneous media.

In this formulation, the occurrence of the so-called spurious, nonphysical modes is predictable. They are unique solutions which do not satisfy the electromagnetic boundary conditions at perfect conductors. A treatment to eliminate such solutions has been suggested.

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A Perturbation Method for the Analysis of Wave Propagation in Inhomogeneous Dielectric Waveguides with Perturbed Media

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Abstract—**This paper presents a perturbation method for determining the modes and the propagation constants of TE and TM waves in inhomogeneous dielectric waveguides whose index distributions depart from well-known profiles; e.g., a parabolic profile for which exact solutions can be obtained. Applying the variable-transformation technique to the wave equations, the wave-equation problem is transformed into the related-equation problem. The approximate solutions of the wave equations are obtained solving the related equation. The method is applied to the analysis of lower order mode propagation in a near-parabolic-index medium. The first-order field functions and the second-order propagation constants are given.**

I. Introduction

The problem of studying the behavior of electromagnetic waves in inhomogeneous media has been of great interest chiefly from mathematical and physical standpoints [1]–[6]. Later a number of methods [7]–[10] were developed to analyze this problem or the equivalent quantum-mechanics problem, most of which are based on the asymptotic expansion method [11] analogous to the Wentzel [3]–Kramers [6]–Brillouin [4], [5] (WKB) method, and these methods have been found to be very useful for weak inhomogeneities.

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Recently, a great variety of refractive-index distributions were used to realize self-focusing optical waveguides. Some of these distributions are not weakly inhomogeneous; the index variations within the distance of a wavelength are relatively rapid. In applying such media to single or quasi-single mode waveguides, it is necessary to analyze the propagation characteristics of lower order modes by suitable methods.

Kurtz and Streifer [8] have applied McKelvey's asymptotic method [7] to the problem of lower order mode propagation, and have found the solutions inaccurate near the center axis of the waveguide. Even if higher order asymptotic approaches are taken into account, it is impossible to improve the accuracy of the solutions near the center axis [11]. To avoid this defect, many authors [12] have used the variational method with the aid of a computer. However, computational labor will be required for the straightforward calculations [13], [14].

In this paper, an analytic method is presented to determine the transverse field functions and the propagation constants of TE and TM waves subjected to lower order mode propagation in inhomogeneous media. The method is based on two techniques. The one is the variable-transformation technique initially presented in nonuniform transmission-line problems by Berger [15] and later transferred to the equivalence problem of lenslike media by Yamamoto and Makimoto [16]. The other is the related